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SPHERICAL WAVES AND SHOCKS

by

G. B. Whitham

Part I: A General Theory of the Propagation of Weak Shocks

Part II: The Kirkwood-Bethe Theory for the Propagation of Shocks in Water.

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1. Introduction

Two theories of the propagation of spherical shocks, having a similar approach, have been put forward independently. One theory was developed by Kirkwood and Bethe in 1942, but then it was contained in a confidential report, and the first published account appeared (without full details) in 1948 in R. H. Cole's book "Underwater Explosions". Unaware of this work, the author obtained a theory, early in 1951, which a recent investigation has shown to have much in common with the Kirkwood-Bethe theory. Kirkwood and Bethe considered the problem of an explosion in water and found how the head shock propagates; it is relevant to note here that their results are intended to hold true (for water) even if the shock is not weak. On the other hand, the author found a theory for the propagation of weak spherical shocks (in any medium) but obtained a description of the whole flow; for example, later shocks which occur in addition to the head shock were predicted and determined. Thus the theories overlap in that they both apply to a weak head shock; the Kirkwood-Bethe theory treats, in addition, a non-weak head shock in water (an addition of much practical value), whilst the author's theory determines, in addition, the later flows and additional shocks.

However, Kirkwood and Bethe made various assumptions, the validity of which was not clear. Now, when their theory is reconsidered in the light of the later theory of weak shocks and derived in a similar way, it becomes apparent how the extension to a non-weak head shock in water may be achieved and the assumptions involved (although they still cannot be completely justified) become clear and appear reasonable. This relation between the two theories is brought out in the present report. First, in Part I the author's theory of weak shocks is described, then in Part II the Kirkwood-Bethe theory is presented (but with the simpler derivation following Part I) as a method of improving the results to apply to a non-weak head shock, in the case when the medium is water. Although it is recommended that the results now be viewed in this way, it should be remembered that it is

the reverse of the chronological order and that most of the results for the head shock in Part I were also obtained (or could have been obtained) in the work of Kirkwood-Bethe.

Part I. The Propagation of Weak Shocks

2. Description of the method

The propagation of weak spherical waves and shocks will be described following the method of Ref. 1, in which the shocks occurring in steady supersonic flow past an axisymmetrical body were determined. The direct analogy to the latter problem is the propagation of cylindrical shocks; but the case of spherical symmetry is similar. The starting point of this method is the linearized theory, which in this case is the simple acoustic theory. The linear results, although extremely valuable for certain purposes, become inadequate to describe the propagation after some time since they do not give the correct "spreading out" of the wave; the linearized theory does not allow for the variation in the local sound speed or for the connection of sound with the fluid, and as time goes on these small variations accumulate. Moreover, shocks, which play an important part in the flow, cannot be included in such a theory. However, these defects can be corrected to give a valid description of the non-linear propagation; this is the basis of the present theory.

A hypothesis is made that the improvement of linearized theory may be achieved simply by replacing the linearized characteristics occurring therein by a more (and sufficiently) accurate approximation to the exact characteristics; that is, it is claimed that the linear results show the correct variation of the quantities along the characteristics but do not have the correct curves (in the (r,t) plane) for the characteristics. Since characteristics are the paths of wavelets, this hypothesis means that the propagation speed of linearized theory, which is c_o (the sound speed in the undisturbed fluid), is not a satisfactory approximation to the exact propagation speed c+u (where c is the local

sound speed and u is the fluid velocity) and must be corrected. Hence the first failing of the linear results is immediately rectified. The other, concerning shocks, also disappears because now the characteristics are not fixed parallel curves; in certain regions they run together in compression, predicting shocks, and the requisiteshocks (to prevent the characteristics from intersecting) are determined from the Rankine-Hugoniot shock conditions. In connection with shocks it is necessary to point out that the linearized results and the theory which is deduced from them neglect changes in the entropy (so that the pressure is a function only of the density). This is valid, however, because the entropy jump at a shock is of third order in the strength of the shock; hence for the weak shocks considered here, the entropy changes may be neglected in obtaining the first order results for the flow behind. Further comments on this point may be found in Ref. 1, p. 303.

In addition to the above physical considerations, the hypothesis is substantiated in many ways (they are discussed in Ref. 1 and are not repeated here) and its validity should not be considered to be in doubt.

It is supposed for definiteness that some initial particle path $r = a(t)$, say is given (where r and t are the distance from the center of symmetry and t is the time), and a description of the propagation outside this path is required. The problem could be set up in other ways but this one is found to be very convenient. For example, in an underwater explosion problem, a(t) would be the radius of the gas sphere. Alternatively, $r = a(t)$ could be taken as a prescribed variation with time of the radius of a "spherical piston." Since the theory only applies to weak disturbances it is assumed that the velocity $a'(t)$ at the particle path is small compared to the undisturbed sound speed c_o; the small quantity δ which measures the smallness of the disturbance is defined as $\max\{a'(t)/c_o\}$. If the explosion is not weak the present theory does not apply in the initial stages. But, a real explosion always decays so that the disturbance will be weak at a sufficient distance from the center.

In this region a particle path could be taken (the data for it being obtained by numerical methods, for example) and the theory applied to find the subsequent propagation to infinity.

3. Linearized theory and its improvement

The linearized (acoustic) theory of spherical waves is obtained by neglecting second and higher order terms in the equations of motion; the following results are well-known. If a potential ϕ is introduced such that $u = \partial\phi/\partial r$, then ϕ satisfies

$$\phi_{rr} + \frac{2}{r}\phi_r = \frac{1}{c_o^2}\phi_{tt} \quad , \tag{1}$$

and the pressure p is given in terms of ϕ by

$$p - p_o = -\rho_o \phi_t \quad , \tag{2}$$

where p_o and ρ_o are the undisturbed values of p and the density ρ. The equation of state is assumed to be

$$p = k_1 \rho^\gamma - k_o \quad , \tag{3}$$

where γ, k_o, k_1 are constant*; usually gases are considered and k_o is zero, but for water k_o is important (its value being approximately 3,000 atmospheres). Then the local sound speed c is found by approximating the Bernoulli equation

$$\frac{c^2}{\gamma-1} + \frac{1}{2}\phi_r^2 + \phi_t = \frac{c_o^2}{\gamma-1} \quad ,$$

to give

$$c \doteq c_o - \frac{1}{2}(\gamma-1)\phi_t/c_o \quad . \tag{4}$$

The solution of (1) representing an outgoing wave is

$$\phi = -f(t-r/c_o)/r \quad ;$$

---

* In general k_o and k_1 are functions of the entropy, but entropy changes are negligible here.

hence,

$$u = \frac{f'\left(t - \frac{r}{c_o}\right)}{c_o r} + \frac{f\left(t - \frac{r}{c_o}\right)}{r^2} \quad , \tag{5}$$

$$c = c_o + \frac{1}{2}(\gamma - 1)\frac{f'\left(t - \frac{r}{c_o}\right)}{c_o r} \quad ,$$

$$p - p_o = \rho_o \frac{f'\left(t - \frac{r}{c_o}\right)}{r} \quad ,$$

These results are corrected, as explained in §2, by replacing the linearized characteristic variable $t - r/c_o$, which appears in them by $\tau(t,r)$ where the curves τ = constant are exact characteristic curves (or at least sufficiently good approximations to them). The definition of τ is not complete since it is only required to be constant on a characteristic; it is made unique by defining it to be the value of t where the characteristic meets the particle path $r = a(t)$. Modifying the linear theory in this way, and at the same time replacing the function f by $\int_o^\tau F(\tau')d\tau'$, the results become

$$u = \frac{F(\tau)}{c_o r} + \frac{\int_o^\tau F(\tau')d\tau'}{r^2} \quad , \tag{6}$$

$$c = c_o + \frac{\gamma - 1}{2}\frac{F(\tau)}{c_o r} \quad , \tag{7}$$

$$p - p_o = \rho_o \frac{F(\tau)}{r} \quad . \tag{8}$$

Now $\tau(t,r)$ is determined from the condition that, on a characteristic τ = constant, $dr/dt = u + c$; therefore, from (6) and (7),

$$\frac{dt}{dr} = \frac{1}{u+c} \simeq \frac{1}{c_o} - 2k\frac{F(\tau)}{r} - \frac{\int_o^\tau F(\tau')d\tau'}{c_o^2 r^2} \quad , \tag{9}$$

where $k = \frac{1}{4}(\gamma+1)/c_o^3$. Equation (8) is integrated, under the condition that $t = \tau$ when $r = a(\tau)$, to give

$$(10)\quad t = \frac{r-a(\tau)}{c_o} - 2kF(\tau)\log\frac{r}{a(\tau)} - \frac{\int_o^\tau F(\tau')d\tau'}{c_o^2}\left(\frac{1}{a(\tau)} - \frac{1}{r}\right) + \tau \quad .$$

Of course, since the expressions (6) and (7) are not exact and in (9) terms of second and higher order in u and $c-c_o$ are neglected, $\tau(t,r)$ is still not the exact characteristic variable; however, in contrast to the characteristics of linearized theory, (10) is a <u>valid</u> approximation to the exact characteristic curves, because the terms neglected in it really are small compared to those retained. Linearized theory, for example, neglected the term $-2kF(\tau)\log\frac{r}{a(\tau)}$ in comparison with $t-\frac{r}{c_o}$ on the basis that F is small; but this term becomes infinitely large compared to $t-\frac{r}{c_c}$ both when $r \to \infty$ and when $t-\frac{r}{c_o}$ is near zero! Equations (6), (7), (8) and (10) form the corrected solution of the flow.

The arbitrary function $F(\tau)$ which occurs in the solution must be determined from the given conditions on the initial curve $r = a(t)$. Since $r = a(t)$ is a particle path, $u = a'(t)$ on it; hence, from (6) remembering that on $r = a(t)$, $t = \tau$,

$$(11)\qquad a'(t) = \frac{F(t)}{c_o a(t)} + \frac{\int_o^t F(t')dt'}{a^2(t)} \quad .$$

An integrating factor for this equation is $c_o a\mu$, where $\mu(t) = \exp\int c_o dt/a(t)$, whence

$$(12)\qquad \int_o^t F(t')dt' = \frac{c_o}{\mu(t)}\int_o^t a(\gamma)a'(\gamma)\mu(\gamma)d\gamma \quad ,$$

$$(13)\qquad F(t) = c_o a(t)a'(t) - \frac{c_o^2}{a(t)\mu(t)}\int_o^t a(\gamma)a'(\gamma)\mu(\gamma)d\gamma \quad .$$

If, in addition to knowing the particle path, the value of p on

it is also known as a function of time, so that $p-p_o = P(t)$ say, then F is determined more simply from (8) as

$$F(t) = \rho_o^{-1} a(t) P(t) \quad . \tag{14}$$

In the special case when $a(t) \equiv 0$ for $t < 0$, and $a''(t)$ is continuous for $t > 0$, a simpler expression than (13) for F may be obtained; the second term in (11) dominates the first (since the ratio of the first term to the second is $O(a(t)/(c_o t)) = O(a'(t)/c_o) = O(\delta)$) and F may be taken as

$$F(t) = \frac{d}{dt}(a^2 a') \quad . \tag{15}$$

For a real explosion, p returns to p_o and u returns to 0 after an infinite time; hence from (6) and (8) it is clear that $F(\tau) \to 0$ as $\tau \to 0$ and that

$$\int_o^\infty F(\tau)\,d\tau = 0 \quad . \tag{16}$$

Later, this result will be shown to have very important consequences. It may also be deduced from (13) by assuming that a(t) is bounded and that $a'(t) \to 0$ as $t \to \infty$ (assumptions which would be true for a real explosion) but the simple argument above is more instructive. In general, the fluid is pushed outward suddenly so that $a'(t)$ is discontinuous and positive at $t = 0$; hence from (13) $F(\tau)$ is discontinuous and positive at $= 0$. The simplest function $F(\tau)$ would be when $a'(t)$ has an initial positive value $c_o \varepsilon$, say (where ε is small and $O(\delta)$), and tends monotonically to zero as $t \to \infty$. Then $F(\tau)$ has the initial positive value $c_o^2 a_o \varepsilon$, where $a_o = a(0)$, and is of the form shown in figure 1, the areas are as above and below the axis being equal by (16).

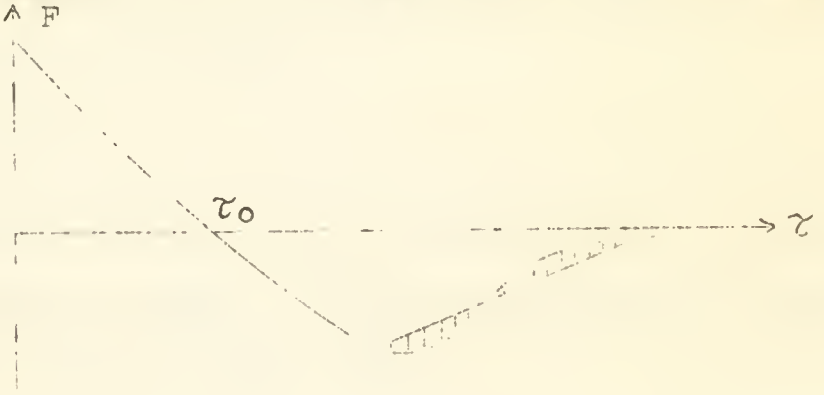

Figure 1.

4. The shocks

The solution obtained in the last section is not yet complete. The characteristics now depend on the solution; where $F'(\tau) < 0$, the characteristics* (10) diverge in expansion, and where $F'(\tau) > 0$ they run together and eventually overlap. The extreme cases of "centered" expansion or compression, with a fan of characteristics from a single point on $r = a(t)$, occur when $F(\tau)$, due to a discontinuity in $a'(t)$, is discontinuous (since in that case a whole range of values of F, and hence a fan of characteristics (10), is found for the same value of τ). If the characteristics were allowed to overlap, the solution would cease to be single-valued and there would be a fold in the flow plane; a shock must be fitted in such regions so that the characteristics meet the shock and are cut off by it before they meet each other. Thus shock formation is predicted in the regions where $F'(\tau) > 0$ or $F(\tau)$ has a positive discontinuity; in the latter case the shock is formed immediately at the particle path $r = a(t)$ (as must be the case since $a'(t)$ would be discontinuous there). For the problem of interest in explosions, $F(\tau)$ has been seen to have a positive discontinuity at $\tau = 0$; hence there will be a shock. This head shock is now determined.

(i) The head shock

The Rankine-Hugoniot shock conditions, expressing the laws of conservation of mass, momentum, and energy through the shock, may be written (assuming that the equation of state is (3) with k_0, k_1 functions of the entropy,

$$u = \frac{1}{\gamma}\frac{c_0^2}{U}\xi \quad , \tag{17}$$

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* For many purposes the term $c_0^{-2}\int_0^{\tau} F(\tau')d\tau'(a^{-1}-r^{-1})$ may be neglected since it is small. Certainly questions of convergence or divergence of characteristics etc. may be decided from the term $2kF(\tau)\log\frac{r}{a(\tau)}$ alone.

$$(18) \qquad U^2 = c_o^2 \left(1 + \frac{\gamma+1}{2\gamma}\,\xi\right) \quad ,$$

$$(19) \qquad c^2 = c_o^2(1+\xi)\left(1 - \frac{c_o^2}{\gamma U^2}\,\xi\right) \quad ,$$

where U is the shock velocity and ξ is defined by

$$(20) \qquad \xi = \frac{p-p_o}{\rho_o c_o^2 \gamma - 1} \quad .$$

In the case when $K_o = 0$ in (3), ξ is the strength $(p-p_o)/p_o$, otherwise it is the strength multiplied by $p_o/(p_o+K_o)$. For the first order theory, second and higher order terms in ξ may be neglected, so

$$(21) \quad u = \frac{c_o \xi}{\gamma} \quad , \quad U = c_o\left(1 + \frac{\gamma+1}{4\gamma}\,\xi\right) \quad , \quad c = c_o\left(1 + \frac{\gamma-1}{2\gamma}\,\xi\right) \quad .$$

(At the same time it may be remarked that the jump in entropy at the shock is small and $O(\xi^3)$; this is the reason that the change of entropy may be neglected in deriving the solution in Section 3.) Clearly the shock conditions furnish two relations: one relation between the flow quantities behind the shock, and one relation to determine U. It is convenient to choose them, from (21), as

$$(22) \qquad u = \frac{2}{\gamma-1}\,(c-c_o) + O(\xi^2) \quad ,$$

and

$$(23) \qquad U = \frac{1}{2}\,(u+c+c_o) + O(\xi^2) \quad .$$

(Actually the error in (22) is $O(\xi^3)$.)

Equation (22) leads to an important point which also arises in finding the shock from (23). To satisfy (22), it is necessary from (6) and (7) that, at the shock, the second term in (6) is of smaller order than the first; that is, since the ratio of the terms is $O(c_o\tau/r)$, it is required that $c_o\tau/r$ is small on the shock. This is so. For, on the shock, a characteristic of the

disturbed flow meets a characteristic of the undisturbed flow ahead (in fact, (23) shows that in the (r,t) plane the shock bisects the angle between them); on the undisturbed characteristic, $t-(r-a_o)/c_o$ is equal to a negative constant, therefore in (10)

$$(24)\quad \frac{a(\tau)-a_o}{c_o} + 2kF(\tau)\log\frac{r}{a(\tau)} + \frac{\int_o^\tau F(\tau')d\tau'}{c_o^2}\left(\frac{1}{a(\tau)}-\frac{1}{r}\right) > \tau \quad ,$$

at the shock. Now $(a(\tau)-a_o)/c_o$ and $c_o^{-2}\int_o^\tau F(\tau')d\tau'(a^{-1}-r^{-1})$ are both $O(\tau\delta)$; thus, since $F(\tau) = O(c_o^2 a_o \delta)$, (24) can only be satisfied if $r/a(\tau) > \exp(K c_o\tau/a_o\delta))$ where K is a certain constant. Hence $c_o\tau/r$ is certainly $O(\delta)$, and is actually much smaller. It is proved, then, that the second term in (6) may be neglected at the shock (in agreement with (22)), and, moreover, that its contribution to (10) may also be neglected. However, away from the shock its presence is extremely important; for example, it is entirely responsible for (16) which, as will be seen later, shows that at least one other shock must occur in addition to the head shock.

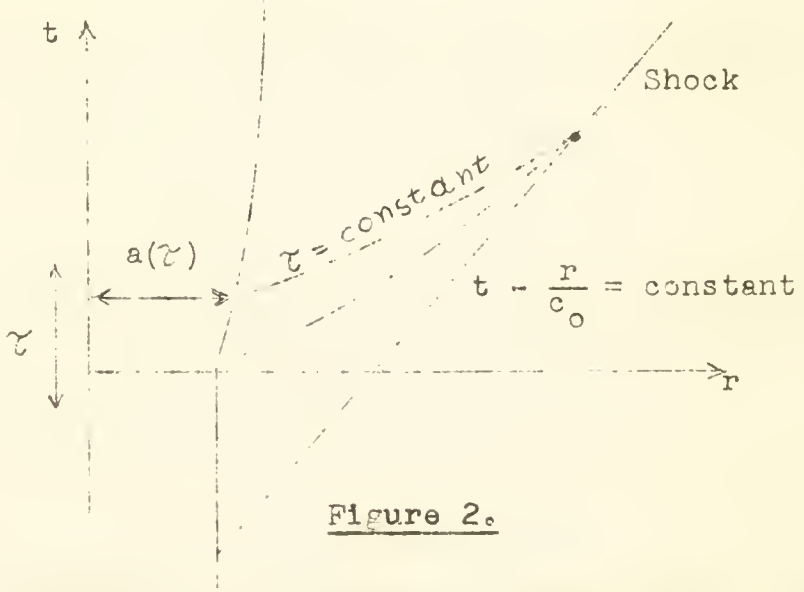

Figure 2.

If the equation of the shock is assumed to be $t = (r-a_o)/c_o - g(r)$ then $U = c_o + c_o^2 g'(r)$ to a first approximation, and using (6) and (7) (with the second term in (6) neglected) (23) gives

$$g'(r) = kF(\tau)r^{-1} \quad . \tag{25}$$

From the argument following (24), (10) may be approximated to

$$t = (r-a_o)/c_o - 2kF(\tau)\log(r/a(\tau)) + \tau \quad , \tag{26}$$

at the shock; hence from the shock equation

$$g(r) = 2kF(\tau)\log(r/a(\tau)) - \tau \quad . \tag{27}$$

In principle equations (25) and (27) determine the function $g(r)$ since τ could be eliminated; it is more convenient, however, to find r as a function of τ i.e. the distance at which the shock cuts the characteristic τ = constant. Differentiating (27) with respect to τ and substituting from (26), the equation for $r(\tau)$ is obtained:

$$\frac{kF(\tau)}{r}\frac{dr}{d\tau} + 2kF'(\tau)\log\frac{r}{a(\tau)} = 1 + \frac{2kF(\tau)a'(\tau)}{a(\tau)} \quad , \tag{28}$$

or,

$$\frac{d}{d\tau}\left\{kF^2(\tau)\log\frac{r}{a(\tau)}\right\} = F(\tau) + \frac{kF^2(\tau)a'(\tau)}{a(\tau)} \quad . \tag{29}$$

Since $\frac{kF(\tau)}{a(\tau)}a'(\tau) = O\left(\frac{u}{c_o}\frac{a'(\tau)}{c_o}\right) = O(\delta^2)$, the second term on the right hand side may be neglected; hence

$$kF^2(\tau)\log\frac{r}{a(\tau)} = \int_o^\tau F(\tau')d\tau' \quad . \tag{30}$$

It may be pointed out that in (30) (and in (26) in the neighborhood of the shock), $a(\tau)$ can be approximated by a_o in the cases when $a_o \neq 0$; $a(\tau)$ itself is only required when $a \to 0$. (This approximation is valid because

$$\frac{\log a(\tau)/a_o}{\log r/a_o} < \frac{\log\left\{1 + O\left(\frac{c_o\delta\tau}{a_o}\right)\right\}}{c_o\tau/(a_o\delta)} = O(\delta^2) \quad ,$$

from the discussion following (24). Hence $\log r/a(\tau)$ may be approximated by $\log r/a_o$.) The values of g and g' are found by

substitution in (25) and (27). The strength of the shock could be found from the shock conditions but is most easily obtained from (8).

First consider the initial velocity and strength of the shock. In the case $a_o \neq 0$, the shock starts at $t = 0$, $r = a_o$, with initial pressure jump $p-p_o = \rho_o F(0)/a_o = \rho_o c_o^2 \varepsilon$ since $F(0) = c_o^2 a_o \varepsilon$, where εc_o is the initial velocity of the "piston curve"; the initial velocity is $U_o = c_o\left\{1+\frac{1}{4}(\gamma+1)\varepsilon\right\}$. Both results agree with the results for plane waves (as they must since the spherical nature of the flow is not apparent initially). In the case $a_o = 0$, (15) shows that $F(\tau) \sim 2c_o^3\varepsilon^3\tau$ for small τ; hence from (30)

$$r \sim c_o \varepsilon \tau \exp\left\{(\gamma+1)^{-1}\varepsilon^{-3}\right\} \quad , \tag{31}$$

from (8), the initial pressure jump is

$$2\rho_o c_o^2 \varepsilon^2 \exp\left\{-(\gamma+1)^{-1}\varepsilon^{-3}\right\} \quad , \tag{32}$$

and from (21)

$$U_o = c_o + \tfrac{1}{2}(\gamma+1)c_o\varepsilon^2 \exp\left\{-(\gamma+1)^{-1}\varepsilon^{-3}\right\} \quad . \tag{33}$$

In the special case of sphere expanding with uniform velocity $c_o\varepsilon$, $F(\tau) = 2c_o^3\varepsilon^3\tau$ and (31)-(33) are true for all r; there is just a single shock of constant strength and velocity given by (32) and (33). In considering the uniformly expanding sphere Lighthill (ref. 2) obtained the result $\log(U/c_o - 1) \sim -(\gamma+1)^{-1}\varepsilon^{-3}$ but did not find the coefficient of the exponential in (33).

For large r, (30) shows that τ is near to τ_o, the first zero of $F(\tau)$ (a zero always exists by (16) for a real explosion; for an artificial example such as the uniformly expanding sphere considered above $\tau \to \infty$ as $r \to \infty$), and (30) may be approximated as $2kF(\tau) = b \log^{-1/2} r/a(\tau_o)$, where

$$b^2 = \frac{\gamma+1}{c_o^3}\int_o^{\tau_o} F(\tau')d\tau' \quad . \tag{34}$$

Therefore $g(r) = b \log^{1/2}(r/a(\tau_o)) - \tau_o$; that is, the shock arrives at the distance r at time

$$(35) \qquad b\sqrt{\log \frac{r}{a(\tau_o)}}$$

earlier than the (asymptotically) undisturbed characteristic $t = (r-a_o)/c_o + \tau_o$ (on which $F = 0$). From (8), the pressure is given by

$$(36) \qquad \xi = \frac{p-p_o}{\gamma^{-1}\rho_o c_o^2} \sim \frac{2\gamma}{\gamma+1} \frac{b}{r \log^{1/2} \frac{r}{a(\tau_o)}}$$

and from (21) the velocity is

$$(37) \qquad U \sim c_o + \frac{c_o b}{2r \log^{1/2} \frac{r}{a(\tau_o)}} \quad .$$

(ii) Later shocks

Condition (16) on F shows that there must always be a second compressive region (for a real explosion) in which $F'(\tau) > 0$ and the characteristics run together to form a shock. The simplest case will be an F-curve as in figure 1; this is considered first. Since $F(\tau)$ is continuous, the distance at which the characteristics overlap to form an envelope is large. For, if (10) is written as $t = h(\tau,r) + \tau$ then the envelope is $\partial h/\partial\tau + 1 = 0$; all the terms in $\partial h/\partial\tau$ except $-2kF'(\tau)\log r/a(\tau)$ are small, therefore on the envelope

$$(38) \qquad \log \frac{r}{a(\tau)} = [2kF'(\tau)]^{-1} = O(\delta^{-1}) \quad .$$

Hence, at the shock, (10) may be approximated by (26) and the second term in (6) neglected.

The shock conditions again give that at any point the shock makes equal angles with the characteristics from either side of it; that is $U = \frac{1}{2}(u_1+c_1+u_2+c_2)$ where subscripts 1 and 2 denote the values on the two sides of the shock. If the shock is $t = (r-a_o)/c_o - g_1(r)$, and characteristics $\tau = \tau_1$, $\tau = \tau_2$ $(\tau_1 < \tau_2)$ intersect on it

$$(40) \qquad g_1'(r) = kr^{-1}\left\{F(\tau_1) + F(\tau_2)\right\} \quad ,$$

and from the equations of the shock and the characteristics

$$(41) \quad g_1(r) = 2kF(\tau_1)\log\frac{r}{a_1} - \tau_1 \quad ,$$

$$= 2kF(\tau_2)\log\frac{r}{a_1} - \tau_2 \quad .$$

(Since the variation of $a(\tau)$ is small and r is relatively large at the shock, $a(\tau)$ has been set* equal to $a_1 \equiv a(\tau_c)$, where τ_c is the common value of τ_1 and τ_2 when they are equal.) To solve (40) and (41), the relation between τ_1 and τ_2, in order that they meet on the shock, is first found. From (41)

$$(42) \quad 2g_1'(r)dr = 2k\left\{F(\tau_1) + F(\tau_2)\right\}r^{-1}dr$$

$$+ \left\{2kF'(\tau_1)\log\frac{r}{a_1} - 1\right\}d\tau_1 + \left\{2kF'(\tau_2)\log\frac{r}{a_1} - 1\right\}d\tau_2$$

and from (40) the terms in dr cancel. From (41), $\log r/a_1$ may be obtained in terms of τ_1 and τ_2 as

$$(43) \qquad \frac{1}{2k\log\frac{r}{a_1}} = \frac{F(\tau_2)-F(\tau_1)}{\tau_2-\tau_1} \quad ,$$

and (42) then becomes

$$(44) \quad \left\{F'(\tau_1)(\tau_2-\tau_1) - \left(F(\tau_2) - F(\tau_1)\right)\right\}d\tau_1$$

$$+ \left\{F'(\tau_2)(\tau_2-\tau_1) - \left(F(\tau_2)-F(\tau_1)\right)\right\}d\tau_2 = 0 \quad .$$

After some manipulation this relation may be written in a form suitable for integration:

$$F(\tau_2)d\tau_2 - F(\tau_1)d\tau_1 = \tfrac{1}{2}\,d\left\{(\tau_2-\tau_1)[F(\tau_2)+F(\tau_1)]\right\} \quad ,$$

whence

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* By the argument following equation (30).

$$(45) \qquad \int_{\tau_1}^{\tau_2} F(\tau)\,d\tau = \tfrac{1}{2}(\tau_2-\tau_1)\{F(\tau_2)+F(\tau_1)\} \quad ,$$

the constant of integration being zero since $\tau_2 = \tau_1$ must be a solution. With the relation between τ_1 and τ_2 known, r and g_1 are given by (43) and (41) in terms of τ_1, say; hence the shock is determined.

Now these shock results are exactly the same as those encountered and discussed in Ref. 1, except that log r/a_1 appears in place of $r^{1/2}$; the following is taken practically word for word from Ref. 1 with appropriate changes in the formulae. The results (45) and (43) have a very simple geometrical interpretation on the F-curve. The left hand side of (45) is the area between the curve, the axis and the two ordinates $\tau = \tau_1$, $\tau = \tau_2$; whilst the right hand side is the area between the same ordinates, the axis and the straight segment joining the points τ_1 and τ_2 of the curve. Hence the condition (45) requires that the "lobes" cut off on each side of the curve by this segment must be equal in area. Moreover from (43) the gradient of this segment is equal to $(2k \log r/a_1)^{-1}$. A typical segment with the lobes shaded is shown in Figure 1.

The path of the shock may be traced by considering the whole series of segments; typical ones 11', 22', 33' are shown in Figure 3. The limiting segment 11', for which the lobe area

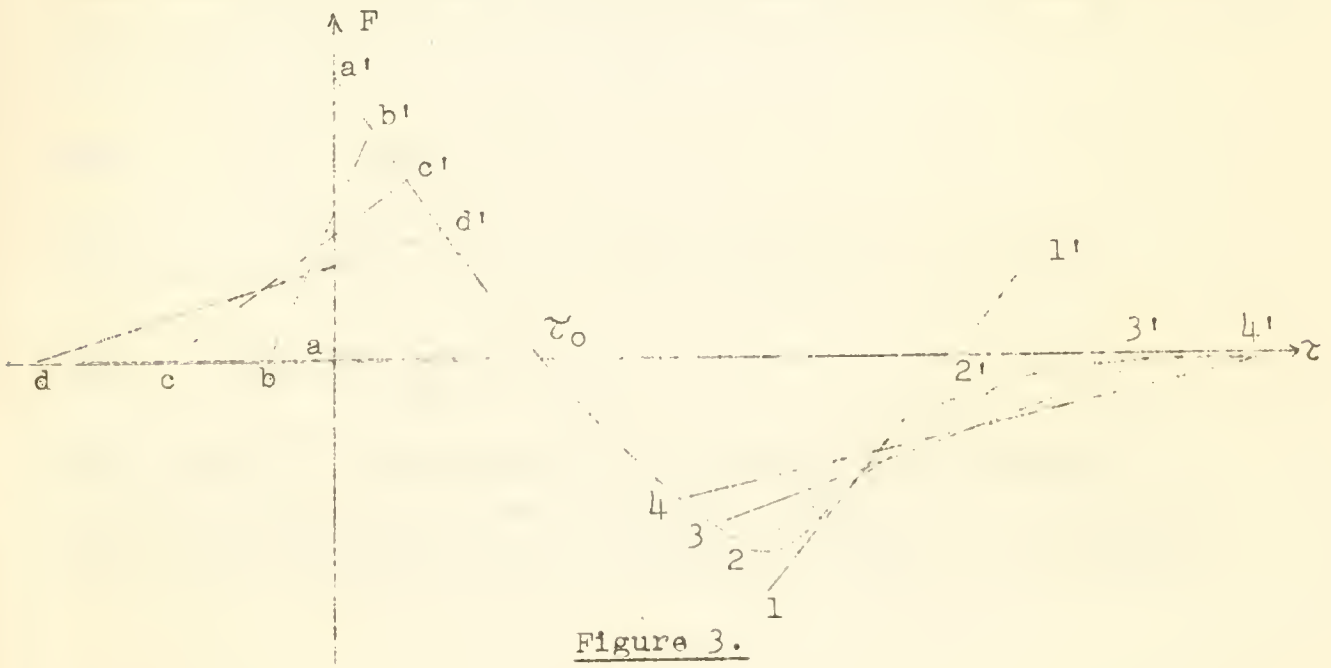

Figure 3.

has decreased to zero, is the tangent to the curve at the point of inflexion $\tau = \tau_c$. Since it has the largest gradient it corresponds to the smallest value of r on the shock; here the shock starts with zero strength (since $F(\tau)$ is continuous at $\tau = \tau_c$). The gradients of the successive segments then decrease, corresponding to increasing r, tending ultimately to be horizontal as $\tau_2 \to \infty$ and $\tau_1 \to \tau_o+$, the zero of $F(\tau)$. From (8) the pressure jump at the shock at any point is

$$(46) \qquad \rho_o \{F(\tau_2) - F(\tau_1)\}/r \quad ;$$

as r increases, the shock first increases in strength and then decays. To study the decay at large distances, the approximations τ_2 large, $\tau_1 - \tau_o$ small may be used. The integral of $F(\tau)$ from 0 to ∞ exists, therefore $F(\tau) = o(\tau^{-1})$ as $\tau \to \infty$; hence (45) may be approximated as

$$(47) \qquad \int_{\tau_o}^{\infty} F(y)dy = \tfrac{1}{2}(\tau_2-\tau_1)(\tau_1-\tau_o)F'(\tau_o) \quad ,$$

and (43) as

$$(48) \qquad 2k \log \frac{r}{a_1} = -\frac{\tau_2-\tau_1}{(\tau_1-\tau_o)F'(\tau_o)} = -\frac{2\int_{\tau_o}^{\infty} F(\tau)d\tau}{(\tau_1-\tau_o)^2F'^2(\tau_o)} \quad .$$

Then from (41), $g_1 = 2k(\tau_1-\tau_o)F'(\tau_o)\log r/a_1 - \tau_o$, and, from (48), $g_1 = -b_1c_o^{-1}\log^{1/2} r/a_1 - \tau_o$, where

$$(49) \qquad b_1^2 = -\frac{\gamma+1}{c_o}\int_{\tau_o}^{\infty} F(\tau)d\tau \quad .$$

That is, the shock arrives at the distance r at a time

$$(50) \qquad b_1 \log^{1/2} \frac{r}{a_1} \quad ,$$

later than the (asymptotically) undisturbed characteristic $t = (r-a_o)/c_o + \tau_o$. It was found in (35) that, at large distances, the head shock was a distance $b \log^{1/2} r/a$, ahead of

the same characteristic. In fact the shocks "balance" since $b = b_1$ by (16).

It would be expected that the more general procedure adopted for the second shock could be applied to the head shock, and indeed this is true. If the F-curve is extended to include the τ-axis for $\tau < 0$, where the characteristics are $t = (r-a_o)/c_o + \tau$, τ = constant, segments cutting off equal lobes of equal area determine the shock, and the slope of the segment is $(2k \log(r/a_o))^{-1}$. Such segments are shown as aa',bb',cc',... in Figure 3; the limiting one lies along the discontinuity at $\tau = 0$, and since its gradient is infinite the corresponding value of r is a_o, i.e. the shock is attached to the initial particle path. The distance increases as the segment gradients decrease, and as $r \to \infty$, the segments tend to the horizontal with $\tau_1 \to -\infty$ and $\tau_2 \to \tau_-$. Thus Figure 3 gives a simple interpretation of the flow pattern. The characteristics $0 < \tau < \tau_c$ be between the shocks; those for which $0 < \tau < \tau_o$ meet the head shock, and those for which $\tau_o < \tau < \tau_c$ the other, the asymptotically straight characteristic, $\tau = \tau_o$, alone reaching infinity. The "segments" give a clear account of the progress of the shocks to infinity.

At large distances between the shocks, τ is near to τ_o; therefore, from (10),

$$t = \frac{r-a(\tau_o)}{c_o} - 2kF(\tau)\log\frac{r}{a(\tau_o)} + \tau_o \quad , \tag{51}$$

approximately. Hence, from (8),

$$\xi = \frac{p-p_o}{\gamma^{-1}\rho_o c_o^2} = \frac{2\gamma}{\gamma+1}\,\frac{r-a(\tau_o)-c_o t+c_o\tau_o}{r\log\frac{r}{a(\tau_o)}} \quad . \tag{52}$$

At the head shock, the value (from (35)) is $2(\gamma+1)^{-1}br^{-1}\log^{-1}r/a(\tau_o)$. in agreement with (36), and at the second shock the value is minus this quantity. Between the shocks ξ falls <u>linearly</u> with t (at fixed r) at a rate

$$\frac{2\gamma}{\gamma+1}\,\frac{c_o}{r\log\frac{r}{a(\tau_o)}} \quad ,$$

which depends only* on γ, c_o and the distance r; it is independent of the particular explosion or initial disturbance considered.

More complicated F-curves than the basic one in Figure 3 may arise, and there may be more than just two shocks. All cases may be described similar to the above and the details may be seen from the account in Ref. 1. They are not repeated here. It is sufficient to remark that in general, there are ultimately only two shocks as $r \to \infty$, since any others eventually run into these two; thus as $r \to \infty$ the behavior of the flow is exactly the same as described above for the F-curve of Figure 3. If $F(\tau)$ has a discontinuity at $\tau = \bar{\tau}$ say, it may be shown that (43), (45) (and hence the "segment" interpretation) and (46) still apply, but near the body, the expression (41) for g_1 is modified. Since the "first segment" is vertical in this case the shock starts with $r = a_1$ (as it should, of course , since a_1 is now the value of $a(\tau)$ at the discontinuity $\tau = \bar{\tau}$). An example with a complicated F-curve, involving more than two shocks, will be given in a later report, in which the shocks produced by the oscillating gas sphere of an underwater explosion will be described.

Part II. The Kirkwood-Bethe Theory for the Propagation of Shocks in Water.

The Kirkwood-Bethe theory will be presented, as explained in the introduction, as a suggested method of modifying the theory of Part I in order to extend its application to a non-weak head shock. The results which will be obtained in this part and the arguments given below are limited to propagation in water, and there is certainly no reason why they should be any improvement over the results of Part I if the medium is air. The explanation of this is that many of the expressions and conditions used in

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* Since r is large, the particular value $a(\tau_o)$ is not important; any typical unit of length in the problem may be inserted.

Part I are true not only for weak shocks and disturbances but for strong ones provided that the medium is water (this is not so for air). In fact the method of derivation will be to follow through the steps of Part I, making only those assumptions which seem reasonable also for a non-weak shock in water.

One of the most important advantages of water from this point of view is that even for quite strong shocks, the change of entropy is small.* Hence, since entropy is conserved on the particle paths behind the shock, the change of entropy may be neglected. Therefore p may be taken as a function only of ρ, the appropriate form being $p = K_1\rho^\gamma - K_0$, with $\gamma \doteq 7$ in the case of water. Then, introducing the velocity potential ϕ, the momentum equation may be integrated to Bernoulli's equation

$$\frac{c^2}{\gamma-1} + \frac{1}{2}u^2 + \phi_t = \frac{c_o^2}{\gamma-1} \quad . \tag{53}$$

In the theory of small disturbances of a compressible fluid presented in Part I, $r\phi_t$ is constant on a characteristic. On the other hand, in the exact theory of an incompressible fluid, $\phi = \Phi(t)/r$ and the characteristics have degenerated into lines t = constant, so that again $r\phi_t$ is constant on a characteristic. In view of these two alternative cases the result that $\phi_t = -F(\tau)/r$, where τ is the characteristic variable, is taken over from Part I, and used even if the disturbance is not weak and the fluid has (small) compressibility. Then Bernoulli's equation (53) becomes

$$\frac{c^2-c_o^2}{\gamma-1} + \frac{1}{2}u^2 = \frac{F(\tau)}{r} \quad . \tag{54}$$

To complete the solution another expression involving u and/or c is required. The expression in Part I for $u(=\phi_r)$, unlike that for ϕ_t, has not the form that u would have in the

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* Actually it is only necessary to neglect the entropy in obtaining Bernoulli's equation ((53) below) and in fact the error in (53) is still small even when the entropy change starts to be appreciable. In Ref. 3, the error is estimated to be 5°/o when the pressure behind the shock is 20,000 atmospheres.

exact compressible fluid theory; the first term in (6) would no longer appear, yet it was found that this first term dominates the second term in the neighborhood of the shock. Hence (6) is not suitable. However, the previous remark, that only the first term in (6) was important near the shock, leads to a valuable suggestion. This first term in the expression for u is

$$(55) \qquad \sigma \equiv \frac{2}{\gamma-1}(c-c_o)$$

in the weak shock theory; we now ask if u may be approximated by σ even in the neighborhood of a non-weak shock. The shock conditions show that this is so; values of u/c_o and σ/c_o are shown in the Appendix for values of ξ ranging from 0 to 80/3, (for water ξ is more relevant than $(p-p_o)/p_o \doteq 3{,}000\ \xi$ as a measure of the shock intensity), and it may be verified that the approximation is good. Moreover if the problem were that of plane waves instead of spherical waves, it would be true that $u = \sigma$ to the same order of approximation <u>throughout</u> the flow . (The Riemann variable $u - 2c/\gamma-1$, and therefore $u-\sigma$, would be constant on each "negative" characteristic; since $u-\sigma$ is approximately zero at the shock, it would follow that $u-\sigma$ is zero to this order of approximation everywhere.) Thus, even for non-weak shocks, $u \doteq \sigma$ at the shock and also for some distance behind (since the spherical nature of the flow is not apparent immediately and it behaves, at first, like one-dimensional flow). But, one important point, which is easily overlooked, remains: In determining the characteristic variable τ (which is used in finding the shock), it is necessary to integrate the equation $dt/dr = (u+c)^{-1}$ along the whole characteristic, even in regions away from the shock where it is not true that $u \doteq \sigma$. Now, in Part I, the same thing occurred, but it was still found that the contribution to this integration from the second term in (6) (the deviation of u from σ in that case) could be neglected in determining the shock, although it would be important in other regions. (This was the result deduced after (24).) <u>It is assumed that this is still true in the case of a non-weak shock</u>. The author feels that this

assumption, that the integrated effect of $u-\sigma$ is negligible at the shock, is the only one in the foundation of the Kirkwood-Bethe theory which can be seriously questioned; it is the only one taken over from Part I which has no additional support from "non-weak results." Nevertheless its use, until it is refuted by experiment or numerical calculations or supplanted by a better theory, seems very reasonable.

The solution for $\sigma(r,\tau)$, obtained from (54) and (55) with $u = \sigma$ is

$$c_o\sigma(1 + \beta\sigma) = \frac{F(\tau)}{r} \quad , \tag{56}$$

where

$$\beta = kc_o^2 = \frac{1}{4}(\gamma+1)/c_o \quad . \tag{57}$$

The characteristic variable $\tau(r,t)$ occurring here, is found from the condition that on τ = constant, $dr/dt = u+c$. With $u = \sigma$, (55) gives $u+c = c_o(1+2\beta\sigma)$; therefore

$$t = \int_{a(\tau)}^{r} \frac{dr}{c_o\{1+2\beta\sigma(r,\tau)\}} + \tau \quad , \tag{58}$$

τ being defined uniquely as before to be the value of t when $r = a$. Equations (56) and (57), giving σ and t as functions of r and the parameter τ, provide the solution to the flow; u is approximately equal to σ near the shock, and p is given in terms of σ by

$$\frac{p-p_o}{p_o+K_o} = \left(\frac{c}{c_o}\right)^{2\gamma/\gamma-1} - 1 = \left(1 + \frac{\gamma-1}{2}\frac{\sigma}{c_o}\right)^{2\gamma/\gamma-1} - 1 \quad . \tag{59}$$

The arbitrary function $F(\tau)$ depends on the boundary conditions on $r = a(\tau)$. In this case $F(\tau)$ cannot be determined from equating u to $a'(\tau)$ on $r = a(\tau)$, since an accurate expression for u in this region is not known; but if, for example, p is given as a function of τ, $F(\tau)$ can be found from (59) and (56) with $r = a(\tau)$.

The shock relations, (22) and (23), which were used for weak shocks, are good approximations also for non-weak shocks in water (see Appendix); (22) is already satisfied by taking $u = \sigma$, and the other, which may be written

$$U = c_o(1 + \beta\sigma) \quad , \tag{60}$$

is now used to determine the shock. First, it is convenient to express (58) differently. From (56),

$$c_o(1+2\beta\sigma)\frac{\partial\sigma}{\partial r} = -\frac{F(\tau)}{r^2} = -\frac{c_o^2\sigma^2(1+\beta\sigma)^2}{F(\tau)} \quad ,$$

therefore

$$\int_{a(\tau)}^{r} \frac{dr}{c_o(1+2\beta\sigma)} = -F(\tau)\int_{a(\tau)}^{\sigma} \frac{d\sigma}{c_o^2\sigma^2(1+\beta\sigma)^2} \quad ,$$

where $a(\tau)$ is a known function of τ, being the value of σ on $r = a(\tau)$ i.e. the solution of (56) with r replaced by $a(\tau)$. Hence (58) becomes

$$t = \tau - \frac{F(\tau)}{c_o^2}\left\{A(\sigma) - B(\tau)\right\} \quad , \tag{61}$$

where

$$A(\sigma) = \int -\frac{d\sigma}{\sigma^2(1+\beta\sigma)^2} = 2\beta\log\left(\frac{1+\beta\sigma}{\beta\sigma}\right) - \frac{1+2\beta\sigma}{\sigma(1+\beta\sigma)} \quad , \tag{62}$$

$$B(\tau) = A(a(\tau)) \quad .$$

In Part I, it was convenient to determine the function $r = R(\tau)$, giving the distance r of the shock on each characteristic; in the present case it is more convenient to find the value of σ at the shock as a function of τ. Let this function $\sigma(R(\tau),\tau)$ be denoted by $s(\tau)$. When $s(\tau)$ is determined, r is given from (56) as $R(\tau)$, where

$$R(\tau) = \frac{F(\tau)}{c_o s(\tau)\left\{1+\beta s(\tau)\right\}} \quad , \tag{63}$$

and t is given from (61) as a function $T(\tau)$, where

$$(64) \qquad T(\tau) = \tau - \frac{F(\tau)}{c_o^2}\left\{A(s(\tau)) - B(\tau)\right\} \quad .$$

Thus the shock and the values of σ, u, p at the shock will be known with τ as parameter.

The shock velocity is $R'(\tau)/T'(\tau)$; therefore, in (60),

$$(65) \qquad R'(\tau) = c_o(1+\beta s)T'(\tau) \quad ;$$

substituting from (63) and (64), (65) gives the following first order equation for $s(\tau)$:

$$(66) \quad -\frac{\beta F(\tau)}{c_o^2 s(1+\beta s)^3}\frac{ds}{d\tau} + \frac{2\beta}{c_o^2}\left\{\log\frac{1+\beta s}{\beta s} - \frac{3+2\beta s}{2(1+\beta s)^2}\right\}F'(\tau)$$

$$= 1 + \frac{1}{c_o^2}\frac{d}{d\tau}\left\{F(\tau)B(\tau)\right\} \quad .$$

As for the corresponding equation (28) of Part I, (66) may be obtained in a form suitable for integration by multiplying by $F(\tau)$; on integration

$$(67) \qquad \frac{\beta}{c_o^2}\left[F^2(\tau)\left\{\log\frac{1+\beta s}{\beta s} - \frac{3+2\beta s}{2(1+\beta s)^2}\right\}\right]_o^\tau$$

$$= \int_o^\tau F(\tau')d\tau' + \frac{1}{c_o^2}\int_o^\tau F(\tau')\frac{d}{d\tau'}\left(F(\tau')B(\tau')\right)d\tau' \quad .$$

This formula determines $s(\tau)$ and, therefore, the shock. It is convenient, however, to simplify the second integral on the right hand side. After some manipulation, it is found that

$$(68) \qquad I \equiv \frac{1}{c_o^2}\int_o^\tau F(\tau')\frac{d}{d\tau'}\left(F(\tau')B(\tau')\right)d\tau'$$

$$= \frac{1}{2c_o^2}\left\{B(\tau)F^2(\tau) - B(0)F^2(0)\right\} + \frac{1}{2c_o^2}\int_o^\tau F^2(\tau')\frac{dB(\tau')}{d\tau'}d\tau'$$

then, since $dB/d\tau = a^{-2}(1+\beta a)^{-2}da/d\tau$, from (62), and

$$(69) \qquad c_o a(1+\beta a) = F(\tau)/a(\tau) \quad ,$$

I is found to be

$$(70) \quad I = \frac{1}{2c_o^2}\left\{B(\tau)F^2(\tau)-B(0)F^2(0)\right\} + \frac{1}{2}\int_{a_o}^{a(\tau)} a^2(\tau')da(\tau') \quad ,$$

where $a_o = a(0)$. Using this expression, with the value for $B(\tau)$ substituted and the last term integrated by parts, and remembering that $s(0) = a_o$ since at $\tau = 0$ the shock meets the particle path, (67) becomes

$$(71) \quad \frac{\beta F^2(\tau)}{c_o^2}\left(\left\{\log\frac{1+\beta s}{\beta s} - \frac{3+2\beta s}{2(1+\beta s)}\right\} - \left\{\log\frac{1+\beta a}{\beta a} - \frac{3+2\beta a}{2(1+\beta a)}\right\}\right)$$
$$= \int_0^\tau F(\tau')d\tau' - \int_0^\tau a(\tau')a(\tau')\frac{da(\tau')}{d\tau'}d\tau' \quad .$$

Kirkwood and Bethe arrive at this result (although not written in the same form) but then remark that for most purposes certain terms may be omitted. In the notation and derivation adopted here, their suggestion amounts to approximating I (equation (70)) by

$$(72) \qquad J = \frac{1}{2}\frac{B(0)}{c_o^2}\left\{F^2(\tau) - F^2(0)\right\} \quad ;$$

it is difficult to believe that this is a valid approximation. For, when τ is small, $J \sim c_o^{-2}B(0)F(0)F'(0)\tau$, whereas

$$(73) \qquad I \sim \left\{c_o^{-2}B(0)F(0)F'(0) + \frac{1}{2}c_o^{-2}F^2(0)B'(0) + \frac{1}{2}a_o^2a'(0)\right\}\tau$$
$$= \left\{c_o^{-2}B(0)F(0)F'(0)+a_o^2a'(0)\right\}\tau \quad ,$$

from (62) and (69); when $F(\tau) \to 0$ (corresponding to large distances on the shock) $J \to -\frac{1}{2}c_o^{-2}B(0)F^2(0)$, whereas

$$(74) \qquad I \to -\frac{1}{2} c_o^{-2} B(0) F^2(0) + \frac{1}{2} \int_{a_o}^{a(\tau_o)} a^2(\tau') da(\tau') \quad .$$

Now in the special case of a small disturbance with $a_o \neq 0$ the dominant term in $B(\tau)$ is $-1/a$ and $c_o a(\tau) = F(\tau)/a(\tau)$ so that $c_o^{-2} B(0)F(0)F'(0) = -a_o^2 a'(0)\big(1+O(\varepsilon)\big)$, where $\varepsilon = a'(0)/c_o$; hence the additional term in (73) actually cancels the term retained in J. Again, for this case, $-\frac{1}{2} c_o^{-2} B(0)F^2(0) = \frac{1}{2} a_o^2 a_o$, therefore the additional term in (74) is certainly not negligible. Thus it appears that I should not be approximated by J.

The shock relation (71) has been written in a form which is especially convenient for comparison with the corresponding relation (30) of Part I. It is seen immediately that in the case of weak disturbances with s and a small, and $c_o a = F(\tau)/a$, $c_o s = F(\tau)/R$, the left hand side of (71) becomes $\beta c_o^{-2} \log(a/s) = \beta c_o^{-2} \log R/a$; also the second term on the right hand side becomes $\int_o^\tau F(\tau') c_o^{-1} a'(\tau') d\tau'$, and since $a'(\tau)/c_o$ is small this term may be neglected in comparison with $\int_o^\tau F(\tau') d\tau'$. Hence (30) is obtained.

Finally, consider the propagation at large distances. Since $s \to 0$, it is clear from (71) that $\tau \to \tau_o$, where τ_o is the first zero of $F(\tau)$; hence $s(\tau)$ and $a(\tau)$ are small and (71) becomes

$$(75) \qquad \beta c_o^{-2} F^2(\tau) \log R/a = \int_o^{\tau_o} F(\tau') d\tau' - \int_o^{\tau_o} a(\tau') a(\tau') a'(\tau') d\tau' \quad .$$

The only change to the asymptotic formulae of Part I, therefore, is in the value of the constant b, in which $\int_o^{\tau_o} F(\tau') d\tau'$ must be replaced by the right hand side of (75). The variation in pressure behind the shock is again given by (52).

Thus the extension of the results of Part I is quite simple; how effective it is must be decided by numerical work on special cases.

Appendix

Numerical values of u, σ, U for a shock in water are given, for various values of the pressure jump, in Ref. 3. Some of the results are set out here in order that the relations $u \doteq \sigma$ and $U \doteq c_o(1+\beta\sigma)$ at the shock, which were used in the Kirkwood-Bethe theory may be verified. The values taken for the constants γ and c_o are 7.15 and 1465 metres/sec, respectively; ρ_o varies slowly with the entropy, its value ahead of the shock being 3.047 kilobars. The pressure is measured in kilobars and it should be remembered that 1 kilobar is approximately 1000 atmospheres, so that the strength $(p-p_o)/p_o$ is approximately 1,000 times the numbers appearing in the pressure column and ξ is approximately 1/3 of these numbers.

p kilobars	u m/sec	σ m/sec	U m/sec	$c_o(1+\beta\sigma)$ m/sec
10	426	416	2335	2314
20	689	663	2880	2817
30	898	859	3320	3217
40	1075	1030	3690	3566
50	1235	1185	4020	3882
60	1380	1330	4325	4178
70	1510	1465	4610	4454
80	1625	1600	4885	4729

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